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On robust identification in the square and king grids ☆

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Abstract

Fault diagnosis of multiprocessor systems gives the motivation for robust identifying codes. We provide robust identifying codes for the square and king grids. Often we are able to find optimal such codes.

Keywords: Multiprocessor system; Fault diagnosis; Robust identification; Optimal code

1. Introduction

Let $G=(V,E)$ be a graph where V is the vertex set and E is the set of edges. Denote by $d(u,v)$ the (*graphic*) *distance* between vertices u and v, i.e., the number of edges in any shortest path between u and v. For the *ball* of radius r centred at $u \in V$ we use the notation

$$B_r(u)=\{v \in V \mid d(u,v) \leqslant r\}.$$

A non-empty subset of V is called a *code* and its elements are *codewords*. Let C be a code. For any $X \subseteq V$, denote its "codeword neighbourhood" by

$$I_r(G,C;X)=C \cap \left(\bigcup_{x \in X} B_r(x)\right).$$

Whenever the graph G and the code C are known from the context we write $I_r(G,C;X)=I_r(X)$. The *symmetric difference* of sets A and B is denoted by $A \triangle B=(A \setminus B) \cup (B \setminus A)$.

Definition 1. Let r and ℓ be non-negative integers and $C \subseteq V$ a code. If for any two distinct subsets $X \subseteq V$ and $\Gamma \subseteq V$ of cardinality at most ℓ each, we have

$$I_r(G,C;X) \triangle A \neq I_r(G,C;\Gamma) \triangle B$$

for any $A, B \subseteq V$ where $|A|, |B| \leqslant t$, then the code C is called *t-robust* $(r, \leqslant \ell)$-*identifying* (*in* G).

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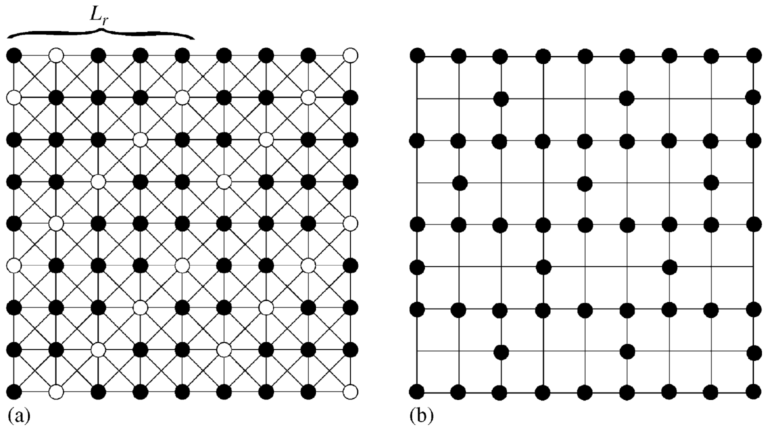

Fig. 1. Optimal constructions (part). Black circles denote the codewords.

Suppose a multiprocessor system is modelled by a graph $G=(V,E)$, that is, a processor corresponds to a vertex and an edge is a bidirectional communication link between two processors in the system. As introduced in [20,12], our task is to locate, in the manner described below, the possible malfunctioning processors assuming that there are at most ℓ of them. Let $X \subseteq V$ denote the set of faulty processors.

We choose a subset of processors, a code C, to perform the following test. Each processor in C checks all the processors within distance r from it and transmits an alarm signal if it detects at least one malfunctioning processor in its neigbourhood, and is silent otherwise. The processors provide thus only one bit of information each. Moreover, we want the system to be more robust meaning that even if some of the transmitted signals are lost and some false alarms received, we must still be able to find the malfunctioning processors. We assume that the number of lost and false alarms together is at most t. Thus, we need to determine X knowing only $I_r(G,C;X)\triangle A$ where $|A|\leqslant t$. This can be done if C is t-robust $(r, \leqslant \ell)$-identifying.

Identifying codes (the case $t=0$) were introduced in [20] and robust identifying ones in [12,23,26]. The underlying graphs that have been mostly studied are the infinite square grid, the infinite triangular grid, the infinite king grid, the infinite hexagonal mesh and binary hypercubes; consult, e.g., [2–16,18,19,21–25].

The codes in Definition 1 are sometimes [24] called *vertex-robust* codes instead of just robust codes. The codes given in Definition 1 are closely related to (somewhat weaker) codes of [12, Definition 2] (see also [26]) discussed thoroughly in [12].

It is convenient to notice [26,24] that a code $C \subseteq V$ is t-robust $(r, \leqslant \ell)$-identifying if and only if

$$|I_r(X)\triangle I_r(\Gamma)|\geqslant 2t+1 \tag{1}$$

for all distinct sets $X \subseteq V$ and $\Gamma \subseteq V$ of cardinality at most ℓ.

In this paper, we restrict ourselves to the cases $t>0$ and $r>0$, and the examined graphs are the following two infinite graphs:

- *The square grid*, where $V=\mathbb{Z}^2$ and two vertices are adjacent if their Euclidean distance equals 1 (see Fig. 1(b)). Denote by Q_n the set of vertices $(i,j)\in\mathbb{Z}^2$ with $|i|\leqslant n$ and $|j|\leqslant n$. Clearly, $|Q_n|=(2n+1)^2$. The *density* of a code $C\subseteq\mathbb{Z}^2$ is defined to be
$$D(C)=\limsup_{n\to\infty}\frac{|C\cap Q_n|}{|Q_n|},$$
- *the king grid*, where $V=\mathbb{Z}^2$ and two vertices are adjacent, if their Euclidean distance equals 1 or $\sqrt{2}$ (see Fig. 1(a)). The density of a code in the king grid is defined exactly as in the square grid.

In the above-mentioned application of robust identification to fault diagnosis in multiprocessor systems, it is natural to try to find a code whose density is as small as possible—in which case we call the code *optimal*.

We will often need the following theorem from [17].

Theorem 1. *Assume that $C \subseteq \mathbb{Z}^2$ is a code in the infinite square (or king) grid. Let $S = \{s_1, s_2, \ldots, s_k\}$ be a subset containing k different vertices. If for all $v \in \mathbb{Z}^2$ we have $|(v + C) \cap S| \geqslant m$, then the density D of C satisfies $D \geqslant m/k$.*

In the sequel, we shall use the following rounding function. For a real number x, the value of the function $[x]$ is defined to be the integer n such that $n - \frac{1}{2} \leqslant x < n + \frac{1}{2}$. It is easy to check that $[x + m] = [x] + m$ for any $m \in \mathbb{Z}$ and for any real number x. Also $[x] \leqslant [y]$ if $x \leqslant y$.

We also denote $I_r(\{v_1, \ldots, v_s\}) = I_r(v_1, \ldots, v_s)$ and if $r = 1$ we omit the subscript.

2. The king grid

In this section, we consider the king grid and assume that ℓ, r and t are all positive integers. We provide optimal densities of t-robust $(r, \leqslant \ell)$-identifying codes whenever they exist. An up-to-date table of the corresponding results for regular identification (i.e., when $t = 0$) can be found in [15].

First we give optimal codes when $\ell = 1$.

Theorem 2. *Let $0 < t \leqslant 2r$. The smallest possible density of a t-robust $(r, \leqslant 1)$-identifying code in the king grid equals $(2t + 1)/(4r + 2)$. If $t > 2r$, there does not exist a t-robust $(r, \leqslant 1)$-identifying code.*

Proof. Let C be a t-robust $(r, \leqslant 1)$-identifying code in the king grid. Let $x = (i, j)$ and $\alpha = (i + 1, j)$. Denote $T_1 = \{(i-r, j-r), (i-r, j-r+1), \ldots, (i-r, j+r)\}$ and $T_2 = \{(i+r+1, j-r), (i+r+1, j-r+1), \ldots, (i+r+1, j+r)\}$. Then $B_r(x) \triangle B_r(\alpha) = T_1 \cup T_2$. By virtue of (1), these $4r + 2$ vertices must contain at least $2t + 1$ codewords of C. Therefore, if $t > 2r$, there cannot exist a t-robust $(r, \leqslant 1)$-identifying code. Let $0 < t \leqslant 2r$. Choosing, for example, $S = I_r((0, 0)) \triangle I_r((1, 0))$ we obtain by Theorem 1 that the density of C is at least $(2t + 1)/(4r + 2)$.

Next we construct a t-robust $(r, \leqslant 1)$-identifying code $C_{r,t}$ with the claimed density, assuming that $0 < t \leqslant 2r$.

Denote

$$c_s = \left[\frac{s(4r + 2)}{2t + 1} \right].$$

Since $t \leqslant 2r$, we get $c_{s+1} > c_s$ for any $s \in \mathbb{Z}$.

We construct the code $C_{r,t}$ from points in the vertical line

$$P_{r,t} = \{(0, c_s) | s \in \mathbb{Z}\}$$

and from its translates

$$C_{r,t} = \bigcup_{k \in \mathbb{Z}} (P_{r,t} + (k, k)).$$

First we give two facts about the code. Denote a vertical set $R_k(a, j) = \{(a, j), (a, j + 1), \ldots, (a, j + k - 1)\}$ and a horizontal set $R'_k(j, a) = \{(j, a), (j + 1, a), \ldots, (j + k - 1, a)\}$ for any $k, a, j \in \mathbb{Z}$. Whenever convenient we just write R_k and R'_k. The code $C_{r,t}$ satisfies the following:

(i) Any (vertical) set of the form R_{4r+2} contains exactly $2t + 1$ codewords of $C_{r,t}$.
(ii) Moreover, any (vertical) set of the form R_{2r+1} contains at least t codewords of $C_{r,t}$.

To prove (i) and (ii) it is clear enough to consider the claim in $P_{r,t}$, i.e. $R_{4r+2}, R_{2r+1} \subseteq P_{r,t}$. The case (i) holds because $c_s + (4r + 2) = c_{s+2t+1}$ for any $s \in \mathbb{Z}$. To show (ii) consider $R_k(a, j)$ and let s be the largest integer such that for the second coordinate $c_s < j$. Because $c_s + (2r + 1) \geqslant c_{s+t}$, the set R_{2r+1} contains at least t codewords.

By the symmetry with respect to the diagonal with slope -1 passing through the origin, it is easy to see that the facts (i) and (ii) are also true for horizontal sets R'_{4r+2} and R'_{2r+1}, respectively.

From the case (i) we can conclude that the density of $C_{r,t}$ is $(2t+1)/(4r+2)$. In order to show that $C_{r,t}$ is t-robust $(r, \leqslant 1)$-identifying, we must show that $|I_r(x)\triangle I_r(\alpha)| \geqslant 2t+1$ for any two $x, \alpha \in \mathbb{Z}^2$ and that also $|I_r(x)\triangle I_r(\emptyset)| = |I_r(x)| \geqslant 2t+1$ for all $x \in \mathbb{Z}^2$.

First we show that $|I_r(x)| \geqslant 2t+1$. The set $B_r(x)$ consists of $2r+1$ disjoint sets of the form R_{2r+1}, which by (ii) each contain at least t codewords. Since $|I_r(x)| \geqslant (2r+1)t \geqslant 2t+1$, we are done.

Let us consider then $x, \alpha \in \mathbb{Z}^2$, $x \neq \alpha$. If $d(x, \alpha) \geqslant 2$, then $B_r(x)\triangle B_r(\alpha)$ contains at least four disjoint sets of form R_{2r+1} or R'_{2r+1}. Since each of them has at least t codewords, $|I_r(x)\triangle I_r(\alpha)| \geqslant 4t \geqslant 2t+1$ and we are done.

Hence it suffices to assume $d(x, \alpha) = 1$.

(1) Suppose first that $x=(i, j)$ and $\alpha=(i+1, j)$. We know from the beginning of the proof that $B_r(x)\triangle B_r(\alpha)=T_1 \cup T_2$. The set $T_2 \cap C$ is obtained by shifting the codewords of

$$T_1' = \{(i-r, j-3r-1), (i-r, j-3r), \ldots, (i-r, j-r-1)\}$$

by $(2r+1, 2r+1)$. Since $T_1 \cup T_1'$ constitute a set of the form R_{4r+2}, by (i) the set $T_1 \cup T_1'$ and hence also $T_1 \cup T_2$ contains at least $2t+1$ codewords and we are done. (The case $\alpha = (i-1, j)$ is the same case.)

(2) If $x=(i, j)$ and $\alpha=(i, j+1)$ (or if $\alpha=(i, j-1)$), then the same argument works (just change the role of vertical sets to horizontal ones).

(3) Assume finally that $x = (i, j)$ and $\alpha = (i+1, j+1)$ (the same argument applies to the cases $\alpha \in \{(i+1, j-1), (i-1, j-1), (i-1, j+1)\}$ as well). Now the set $B_r(x)\triangle B_r(\alpha)$ consists of two sets T_1 and $T_2 + (0, 1)$ of type R_{2r+1} and of two sets $T_3 = \{(i-r, j-r), (i-r+1, j-r), \ldots, (i+r, j-r)\}$ and $T_4 = \{(i-r+1, j+r+1), (i-r+2, j+r+1), \ldots, (i+r+1, j+r+1)\}$ of type R'_{2r+1}. These sets intersect in exactly two points $v_1 = (i-r, j-r)$ and $v_2 = (i+1+r, j+1+r)$. Thus, $I_r(x)\triangle I_r(\alpha)$ contains at least $4t-2$ codewords and since $4t-2 \geqslant 2t+1$ for $t > 1$, we are done unless $t = 1$.

Let $t = 1$. If v_1 or v_2 does not belong to the code, then $|I_r(x)\triangle I_r(\alpha)| \geqslant 3$ and we are done. Suppose then that both v_1 and v_2 are in the code. To show that T_1 gives a third codeword, consider $R_{2r+1}(0, c_s) \subseteq P_{r,1}$ (this corresponds to T_1 where v_1 corresponds to the codeword $(0, c_s)$ for some s). The fact that $c_{s+1} \leqslant c_s + 2r$ shows that there must be also another codeword in T_1. Hence $|I_r(x)\triangle I_r(\alpha)| \geqslant 3$ and this completes the proof. □

Because there does not exist a 0-robust $(r, \leqslant \ell)$-identifying code for $\ell \geqslant 3$ (see [14]), there does not exist a t-robust $(r, \leqslant \ell)$-identifying code for $t > 0$ either if $\ell \geqslant 3$. So it is enough to study the case $\ell = 2$. Moreover, because the symmetric difference $(B_r(x+r, y+r) \cup B_r(x+r+1, y+r+1))\triangle(B_r(x+r, y+r+1) \cup B_r(x+r+1, y+r))$ equals

$$\{(x, y), (x+2r+1, y), (x, y+2r+1), (x+2r+1, y+2r+1)\}$$

for $(x, y) \in \mathbb{Z}^2$, we immediately see that if $t \geqslant 2$ there does not exist a t-robust $(r, \leqslant 2)$-identifying code. Next we consider the case $\ell = 2$ and $t = 1$.

Theorem 3. *Let $r > 0$. A code C is 1-robust $(r, \leqslant 2)$-identifying in the king grid, if and only if for all $(x, y) \in \mathbb{Z}^2$, the sets*

$$\{(x, y), (x+2r+1, y), (x, y+2r+1), (x+2r+1, y+2r+1)\},$$

$$\{(x, y), (x, y+1), (x, y+2), \ldots, (x, y+2r)\}$$

and

$$\{(x, y), (x+1, y), (x+2, y), \ldots, (x+2r, y)\}$$

each contain at least three codewords of C.

Proof. This is a straightforward extension of Theorem 2.1 from [14]. □

Theorem 4. *The smallest possible density for a 1-robust $(r, \leqslant 2)$-identifying code equals 1 if $r = 1$ and $\frac{3}{4}$ if $r \geqslant 2$.*

Proof. In the case $r=1$ of Theorem 3 gives that $C=\mathbb{Z}^2$ is 1-robust $(1, \leqslant 2)$-identifying and its density is clearly 1. Moreover, the second set of Theorem 3 consists of three points, so the density of such a code is bounded below by 1 according to Theorem 1 (take, for instance, $S=\{(0,0),(0,1),(0,2)\}$).

Let then $r \geqslant 2$. Let $P=\{(0,j) \mid j \not\equiv 3 \bmod 4\}$ and

$$L_r = \bigcup_{k=0,1,2,\ldots,2r} (P+(k,k)).$$

We show that the code (see Fig. 1(a) in the case $r=2$)

$$C_r = \bigcup_{v\in\mathbb{Z}} (L_r + v(2r+1,2))$$

is 1-robust $(r, \leqslant 2)$-identifying using Theorem 3. Clearly, $D(C_r)=\frac{3}{4}$.

In the set $\{(x,y),(x+2r+1,y),(x,y+2r+1),(x+2r+1,y+2r+1)\}$, the first and the third point are on the same vertical line and the other two are on another at distance $2r+1$. By the construction of C_r, we know that the point $(x+2r+1,y)$ (resp., the point $(x+2r+1,y+2r+1)$) belongs to the code if and only if the point $(x,y-2)$ (resp., if $(x,y+2r-1)$) does, which we obtain by shifting back by $-(2r+1,2)$. Therefore, it suffices to show that the set $\{(x,y),(x,y-2),(x,y+2r+1),(x,y+2r-1)\}$ contains at least three codewords. Because the set of codewords on each vertical line is a translate of P, these four points actually contain at least three codewords, since the elements in $\{y,y-2,y+2r+1,y+2r-1\}$ belong to distinct residue classes modulo four.

The second (vertical) set of Theorem 3 evidently contains at least three codewords.

Let us look at the third set of Theorem 3. From the set $\{(x,y),(x+1,y),\ldots,(x+2r,y)\}$ at least $r+1$ points belong to $L_r+v(2r+1,2)$ for some $v\in\mathbb{Z}$ and the rest (if any) belongs to the neighbour $L_r+(v+1)(2r+1,2)$ or $L_r+(v-1)(2r+1,2)$. If $r+1\geqslant 4$ we are done, because every four consecutive points on a horizontal line in L_r have at least three codewords. So assume $r=2$. Consider the set $\{(x,y),(x+1,y),(x+2,y),(x+3,y),(x+4,y)\}$. Notice that each horizontal line of L_2 consists of five points. Thus, we can find v such that there is at least three points of the set in $L_2+v(5,2)$. If there is at least four points, we are done. So assume that exactly three of the five points belong to the $L_2+v(5,2)$. Then $L_2+(v+1)(5,2)$ or $L_2+(v-1)(5,2)$ has exactly two of the five points. Then $L_2+v(5,2)$ provides at least two codewords and the neighbour at least one more. $\square$

The corresponding result for $t=0$ can be found in [14].

3. The square grid

We continue by looking at the square grid with $t>0$, $r>0$ and $\ell>0$. An up-to-date table concerning the results of the regular identification (that is, $t=0$) can be found from [15] with one exception, namely, the optimal density of $(1, \leqslant 1)$-identifying code in the square grid was recently proven to equal $\frac{7}{20}$ in [1].

Let us first look at the case $t=r=\ell=1$. If C is a 1-robust $(1, \leqslant 1)$-identifying code in the square grid, then $|I(x)\triangle I(\emptyset)|=|I(x)|\geqslant 3$ for all $x\in\mathbb{Z}^2$. Denote $C_i=\{c\in C \mid |I(c)|=i\}$, $N=\mathbb{Z}^2\setminus C$ and $N_i=\{x\in N \mid |I(x)|=i\}$. Consequently, $C_1=C_2=\emptyset$ and $N_0=N_1=N_2=\emptyset$.

Next we introduce a voting scheme (a counting device, see, e.g., [18]) utilized in the proof of Theorem 5.

Elements in the sets N_4, C_4 and C_5 give votes to the elements of N_3 and C_3 according to the following rules:

- A point $v\in N_4$ gives half a vote to every point u at Euclidean distance $\sqrt{2}$ if $u\in N_3$ or $u\in C_3$.
- Let $x\in C_4$ and denote the only non-codeword in $B_1(x)$ by y. Now x gives $\frac{1}{2}$ votes to each codeword $c\in I(x)$ provided that $c\neq x$ and $c\in C_3$. The point x also gives $\frac{1}{2}$ votes to each of the points u which are at Euclidean distance $\sqrt{2}$ from x provided that $u\notin B_1(y)$ and $u\in C_3\cup N_3$.
- A point $x\in C_5$ gives $\frac{1}{2}$ votes to all the points $u\in C_3\cup N_3$ which are at Euclidean distance $\sqrt{2}$ or 1 from x.

Lemma 1. *Assume that C is a 1-robust $(1, \leqslant 1)$-identifying code. Then each point in $N_3\cup C_3$ gets at least one vote. Moreover, elements in N_4 and C_4 give at most two votes each and elements in C_5 give at most four votes each.*

Proof. Let $x=(i,j)\in N_3$ and we may assume that $I(x)=\{(i,j-1),(i-1,j),(i,j+1)\}$. Then $\alpha=(i-1,j+1)$ must be in $N_4\cup C_4\cup C_5$, because $|I(x)\triangle I(\alpha)|\geqslant 3$. By symmetry, also $(i-1,j-1)\in N_4\cup C_4\cup C_5$. Thus, x gets half a vote from both of them according to the rules.

Let then $x=(i,j)\in C_3$. There are two types of positions for the codewords (up to rotations). Let first $I(x)=\{x,(i,j-1),(i,j+1)\}$. Then $\alpha=(i,j+1)$ belongs to $C_4\cup C_5$ because $|I(x)\triangle I(\alpha)|\geqslant 3$. By symmetry, also $(i,j-1)\in C_4\cup C_5$. According to the rules, x receives $\frac{1}{2}$ votes from both. Assume then that $I(x)=\{x,(i,j+1),(i+1,j)\}$. Now both $(i,j+1)$ and $(i+1,j)$ belong to $C_4\cup C_5$, because $|I(x)\triangle I((i,j+1))|\geqslant 3$ and $|I(x)\triangle I((i+1,j))|\geqslant 3$. They both give half a vote to x. We have now verified the first claim.

The rules immediately imply that the elements in N_4 give at most two and elements in C_5 give at most four votes.

Next we show that a point $x\in C_4$ can give at most two votes—so that not all the five possible points receive votes from x. Without loss of generality, we take $x=(0,0)$ and $I(x)=\{x,(-1,0),(0,1),(1,0)\}$. Assume to the contrary, that all the five possible points $(-1,0)$, $(-1,1)$, $(0,1)$, $(1,1)$ and $(1,0)$ belong to $N_3\cup C_3$ and thus received half a vote each.

If $(1,1)\in C_3$, then $|I((0,1))\triangle I((1,0))|=2$, a contradiction. The same goes to $(-1,1)$, so we assume that both $(-1,1)$ and $(1,1)$ belong to N_3. Consequently, either $(2,1)$ or $(1,2)$ belong to C, and by symmetry either $(-2,1)\in C$ or $(-1,2)\in C$.

If $(2,1)\in N$, then $(2,0)\in C$ (because $|I((2,1))|\geqslant 3$) and hence $(1,-1)\in N$. Similarly, if $(-2,1)\in N$, then $(-1,-1)\in N$. This leads to $|I((0,-1))|<3$, which is impossible, and therefore, it is not possible that both $(2,1)$ and $(-2,1)$ are in N.

Let first $(2,1)\in N$ and $(-2,1)\in C$. If $(-2,0)\in C$, then again $(-1,-1)\in N$ and $|I((0,-1))|<3$. Let $(-2,0)\in N$. But then $|I((-1,0))\triangle I((0,-1))|<3$, a contradiction. By symmetry, the case $(-2,1)\in N$ and $(2,1)\in C$ leads to a contradiction.

Hence, it suffices to check the case $(-2,1),(2,1)\in C$. Now evidently $(0,2)\in C$ and $(-1,2),(1,2)\in N$. But then $|I((0,1))\triangle I((0,2))|<3$, which is not allowed. This gives the sought result because a point in C_4 can give half a vote to at most four points. □

Theorem 5. *The smallest possible density of a* 1-*robust* $(1,\leqslant 1)$-*identifying code in the square grid equals* $\frac{2}{3}$.

Proof. A 1-robust $(1,\leqslant 1)$-identifying code with density $\frac{2}{3}$ is given in Fig. 1(b).

Next we show, utilizing the voting scheme, that the density of a 1-robust $(1,\leqslant 1)$-identifying code C is at least $\frac{2}{3}$.

Let us look at the amount of votes received by N_3 and C_3 in Q_{n-1} (recall the definition of Q_n from Introduction), that is, by $N_3\cap Q_{n-1}$ and $C_3\cap Q_{n-1}$. According to the voting scheme, these votes come from points in $Q_n\cap(N_4\cup C_4\cup C_5)$. The number of votes given by the points in $Q_n\setminus Q_{n-1}$ is at most $4|Q_n\setminus Q_{n-1}|=32n$. Therefore, we can write using the previous lemma, that

$$|N_3\cap Q_{n-1}|+|C_3\cap Q_{n-1}|\leqslant 2|N_4\cap Q_{n-1}|+2|C_4\cap Q_{n-1}|+4|C_5\cap Q_{n-1}|+32n.$$

Because $C=C_3\cup C_4\cup C_5$ and $N=N_3\cup N_4$, we obtain

$$3|N_3\cap Q_{n-1}|+3|C_3\cap Q_{n-1}|\leqslant 2|N\cap Q_{n-1}|+2|C\cap Q_{n-1}|+2|C_5\cap Q_{n-1}|+32n. \tag{2}$$

To obtain the result for the infinite grid we now proceed in the usual way (as, for example, in [8]). Counting in two ways the number of pairs (c,x) such that $c\in C\cap Q_n$, $x\in Q_n$ and $d(c,x)\leqslant 1$ one gets

$$\begin{aligned}5|C\cap Q_n| &\geqslant 4|C\cap Q_{n-1}|-|C_3\cap Q_{n-1}|+|C_5\cap Q_{n-1}|+4|N\cap Q_{n-1}|-|N_3\cap Q_{n-1}|\\ &\geqslant 4|C\cap Q_{n-1}|+4|N\cap Q_{n-1}|+|C_5\cap Q_{n-1}|\\ &\quad -\tfrac{2}{3}(|N\cap Q_{n-1}|+|C\cap Q_{n-1}|+|C_5\cap Q_{n-1}|)-\tfrac{32}{3}n\\ &\geqslant \tfrac{10}{3}|Q_{n-1}|-\tfrac{32}{3}n\end{aligned}$$

using (2) in the second inequality.

This yields

$$\frac{|C \cap Q_n|}{|Q_n|} \geqslant \frac{2}{3} - \frac{112n}{15(2n+1)^2}$$

and this approaches $\frac{2}{3}$ as n grows. Hence the density of C is at least $\frac{2}{3}$. $\square$

Next we concentrate on other values of t and r for $\ell = 1$. Let $(x, y) \in \mathbb{Z}^2$. Denote

$$T_k(x, y) = \{(x, y), (x+1, y+1), \ldots, (x+k-1, y+k-1)\}$$

and

$$U_k(x, y) = \{(x, y), (x+1, y-1), \ldots, (x+k-1, y-k+1)\}.$$

We use the notation $T_k = T_k(x, y)$ (resp., $U_k = U_k(x, y)$) when we want to consider k successive points on a diagonal with slope 1 (resp., -1) and need not to specify (x, y).

Theorem 6. *Let $t, r > 0$ and $t \leqslant 2r$. The smallest possible density D of a t-robust $(r, \leqslant 1)$-identifying code in the square grid satisfies*

$$\frac{2t+1}{4r+2} \leqslant D \leqslant \frac{t+1}{2r+1}$$

provided that $r \geqslant 2$ if $t = 1$.

If $t > 2r$, there does not exist a t-robust $(r, \leqslant 1)$-identifying code.

Proof. Assume that C is a t-robust $(r, \leqslant 1)$-identifying code. Let $x = (i, j)$ and $\alpha = (i+1, j+1)$. The set $B_r(x) \triangle B_r(\alpha)$ is the set $U_{r+1}(i-r, j) \cup U_r(i-r+1, j) \cup U_r(i+1, j+r) \cup U_{r+1}(i+1, j+r+1)$. These $4r+2$ points must contain at least $2t+1$ codewords of C. Therefore, if $t > 2r$, there does not exist a t-robust $(r, \leqslant 1)$-identifying code. Let $t \leqslant 2r$. Theorem 1 gives us the lower bound $(2t+1)/(4r+2)$ on the density of C.

Next we construct a t-robust $(r, \leqslant 1)$-identifying code $C_{r,t}$ provided that $t, r > 0$, $t \leqslant 2r$ and $r \geqslant 2$ if $t = 1$. Denote

$$c_s = \left\lceil \frac{s(2r+1)}{t+1} \right\rceil.$$

We choose

$$C_{r,t} = \{(k, c_s) | s, k \in \mathbb{Z}\}.$$

The code satisfies the following:

(i) Any set of the form $R = \{(a_0, j), (a_1, j+1), \ldots, (a_{2r}, j+2r)\}$, where $j, a_i \in \mathbb{Z}$ for all $i = 0, \ldots, 2r$, contains exactly $t+1$ codewords of $C_{r,t}$.

(ii) Every set of the form T_r or U_r contains at least $\lfloor t/2 \rfloor$ codewords of $C_{r,t}$.

(iii) If $t = 1$, then every set of the form $R' = \{(a_0, j), (a_1, j+1), \ldots, (a_r, j+r)\}$, where $j, a_i \in \mathbb{Z}$ for all $i = 0, \ldots, r$, contains at least one codeword. If $t = 3$, then every such set contains at least two codewords.

The first fact (i) follows because $c_s + (2r+1) = c_{s+t+1}$ for any $s \in \mathbb{Z}$, and the case (ii) holds, since $c_s + r \geqslant c_{s+\lfloor t/2 \rfloor}$ for any s (recall that $t \leqslant 2r$). The fact (iii) goes similarly due to the fact that $c_s + (r+1) \geqslant c_{s+1}$ if $t = 1$ and $c_s + (r+1) \geqslant c_{s+2}$ for $t = 3$.

By (i), we know that the density of $C_{r,t}$ equals $(t+1)/(2r+1)$. We need to check that $|I_r(x) \triangle I_r(\alpha)| \geqslant 2t+1$ and $|I_r(x)| \geqslant 2t+1$ for any $x, \alpha \in \mathbb{Z}^2$, $x \neq \alpha$. First we verify that $|I_r(x) \triangle I_r(\alpha)| \geqslant 2t+1$.

If $d(x, \alpha) \geqslant 3$, then $B_r(x) \triangle B_r(\alpha)$ contains at least six disjoint sets of types U_r, U_{r+1}, T_r or T_{r+1}—indeed, if $x = (i, j)$ and $\alpha = (a, b)$ where $i \leqslant a$ and $j \leqslant b$ (the other cases go similarly) and $d(x, \alpha) \geqslant 3$, then the sets $U_{r+1}(i-r, j)$, $U_r(i-r+1, j)$, $U_{r+1}(i-r+1, j+1)$ from $B_r(x)$ and the sets $U_{r+1}(a, b+r)$, $U_r(a, b+r-1)$, $U_{r+1}(a-1, b+r-1)$ from $B_r(\alpha)$ are (at least) in the symmetric difference. Thus, by (ii), $|I_r(x) \triangle I_r(\alpha)| \geqslant 6\lfloor t/2 \rfloor$. If t is even or if t is odd and $t \geqslant 5$, then $6\lfloor t/2 \rfloor \geqslant 2t+1$ and we are done. Let then $t=1$ or 3. Observe that four of the six sets are of the form U_{r+1} or T_{r+1}. Then (iii) gives that

$$|I_r(x) \triangle I_r(\alpha)| \geqslant \begin{cases} 4 & \text{if } t=1, \\ 10 & \text{if } t=3, \end{cases}$$

and we are done.

Consequently, it suffices to assume that $1 \leqslant d(x, \alpha) \leqslant 2$.

(1) Let first $x=(i, j)$ and $\alpha=(i+1, j)$. In $B_r(x) \triangle B_r(\alpha)$ we can find two disjoint sets of the form R (indeed, $B_r(x) \setminus B_r(\alpha)$ and $B_r(\alpha) \setminus B_r(x)$), and hence there is, by (i), at least $2(t+1)$ codewords in $I_r(x) \triangle I_r(\alpha)$ and we are done. The case $\alpha=(i+2, j)$ goes analogously.

(2) Take $x=(i, j)$ and $\alpha=(i, j+1)$. There are two disjoint sets of the form R in $B_r(x) \triangle B_r(\alpha)$. These are $U_{r+1}(i-r, j) \cup U_r(i+1, j+r)$ and $T_{r+1}(i-r, j+1) \cup T_r(i+1, j-r+1)$. Hence $|I_r(x) \triangle I_r(\alpha)| \geqslant 2t+2$.

(3) Let then $x=(i, j)$ and $\alpha=(i+1, j+1)$. The set $B_r(x) \triangle B_r(\alpha)$ consists of two disjoint sets of type R, namely, $U_{r+1}(i-r, j) \cup U_r(i+1, j+r)$ and $U_r(i-r+1, j) \cup U_{r+1}(i+1, j+r+1)$. Therefore, $|I_r(x) \triangle I_r(\alpha)| \geqslant 2t+2$. The case $\alpha=(i-1, j+1)$ is analogous.

(4) Suppose $x=(i, j)$ and $\alpha=(i, j+2)$. Let $L=\{(a, j+1) | a \in \mathbb{Z}\}$. Notice that $L \cap (B_r(x) \triangle B_r(\alpha))=\emptyset$. We can find three disjoint sets of type R among the elements of $(B_r(x) \triangle B_r(\alpha)) \cup L$. These are, for example, $U_{r+1}(i-r-1, j+1) \cup T_r(i-r, j+2)$, $U_{r+1}(i-r, j+1) \cup T_r(i-r+1, j+2)$ and $T_{r+1}(i+1, j-r+1) \cup U_r(i+1, j+r+1)$. Each of these three sets contains exactly one point which is in L. By (i), we know that $|I_r(x) \triangle I_r(\alpha)| \geqslant 3t$ and we are done.

All the other choices for $\alpha \in B_2(x)$ can be treated using the cases from (1) to (4) above by changing the roles of x and α.

To complete the proof, we must verify that $|I(x)| \geqslant 2t+1$ for all $x \in \mathbb{Z}$. Denote $x=(i, j)$. The set $I(x)$ contains three disjoint sets $\{(i, j-r), (i, j-r+1), \ldots, (i, j+r)\}$, $\{(i-1, j-r+1), (i-1, j-r+2), \ldots, (i-1, j+r-1)\}$ and $\{(i+1, j-r+1), \ldots, (i+1, j+r-1)\}$. First one of these is of the form R and the latter two are also of the form R except that they both are two points short. By virtue of (i), $|I(x)| \geqslant (t+1)+(t-1)+(t-1)=3t-1$. If $t \geqslant 2$ we are done. Suppose then $t=1$. Then we can assume that $r \geqslant 2$. In this case the latter two sets above contain a subset of the form R' and hence $|I(x)| \geqslant 4$. □

For $r=1$ and $t=2$, the fact $|I(x) \triangle I(\emptyset)|=|I(x)| \geqslant 5$ for all $x \in \mathbb{Z}^2$, implies lower bound 1 on the density of any 2-robust $(1, \leqslant 1)$-identifying code, which is better than the lower estimate of the previous theorem. In fact, this lower bound 1 is achieved according to the upper bound of the previous theorem.

We conclude this section by examining t-robust $(r, \leqslant \ell)$-identifying codes in the square grid for $\ell \geqslant 2$. Let t, $r>0$. There does not exist a t-robust $(r, \leqslant \ell)$-identifying code for $\ell \geqslant 3$, because $|B_r((0,0),(2,2)) \triangle B_r((0,0),(1,1),(2,2))| \leqslant 1$.

Therefore, it is enough to deal with the case $\ell=2$. Moreover, there does not exist t-robust $(r, \leqslant 2)$-identifying codes for any $t \geqslant 4$. Indeed, $(B_r((-1,0)) \cup B_r((1,0))) \triangle (B_r((0,-1)) \cup B_r((0,1)))$ equals

$$A=\{(-r-1,0),(-r,0),(0,r+1),(0,r),(r+1,0),(r,0),(0,-r-1),(0,-r)\}$$

and this set has only eight points.

Let $C_{r,t}$ be a t-robust $(r, \leqslant 2)$-identifying code. Since there must be $2t+1$ codewords among every translate of A, Theorem 1 yields $D(C_{r,t}) \geqslant (2t+1)/8$ when $t=1,2,3$. Next we show that asymptotically, with respect to r, the bound $(2t+1)/8$ can be achieved (for a corresponding result on $t=0$ see [15]).

Theorem 7. *Let $r>0$ and $0<t<4$. Assume that $C \subseteq \mathbb{Z}^2$. If for all $(x, y) \in \mathbb{Z}^2$ the sets $T_{r+1}(x, y)$ and $U_{r+1}(x, y)$ contain at least $2t+1$ codewords of C and every translate of the set A contains also at least $2t+1$ elements of C, then C is t-robust $(r, \leqslant 2)$-identifying in the square grid.*

Proof. This is a straightforward extension of the proof of [15, Theorem 2]. □

Theorem 8. *For each $t=1,2,3$, there exist a sequence of t-robust $(r, \leqslant 2)$-identifying codes $(C_{r,t})$ with densities such that $D(C_{r,t}) \to (2t+1)/8$ as $r \to \infty$.*

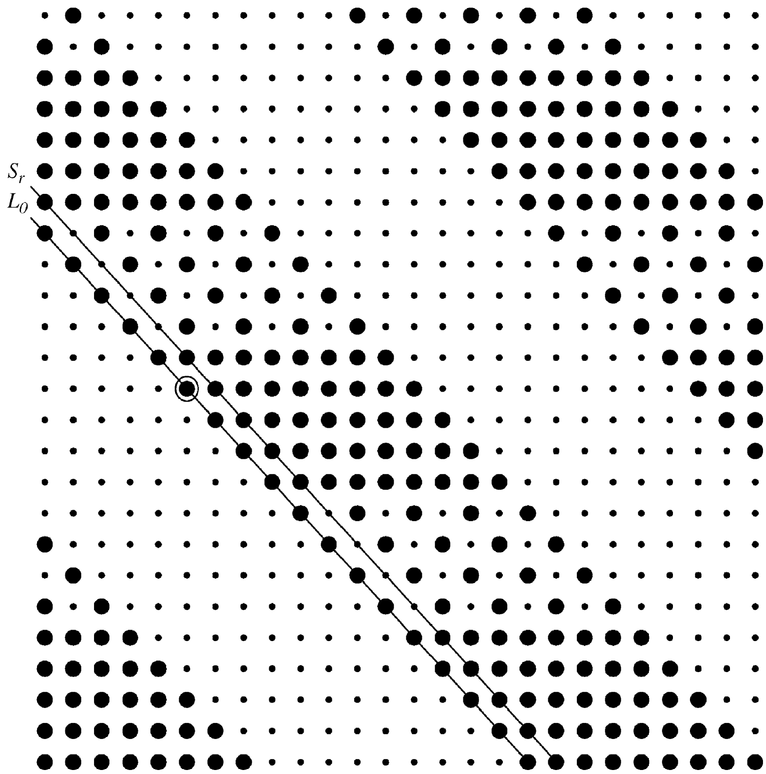

Fig. 2. The code C_4^1. Large black circles denote the codewords and the circled point is the origin.

Proof. We denote a diagonal with slope -1 by

$$L_s = \{(a, s-a) | a \in \mathbb{Z}\}.$$

Denote further

$$S_r = \{(a, 1-a) | a \in \mathbb{Z} \text{ and } a \equiv 0, 1, 2, \ldots, r \bmod 2r+1\}.$$

Case $t=1$: Let (see Fig. 2)

$$H_r = \left(\bigcup_{k=0,\ldots,r-1} ((L_0 \cup S_r) + (2k, 0)) \right) \cup L_{2r}.$$

We show below that the code

$$C_r^1 = \bigcup_{v \in \mathbb{Z}} (H_r + (4r+2)(v, 0))$$

gives to every translate of A at least three codewords. The density

$$D(C_r^1) = \frac{3r^2 + 4r + 1}{2(2r+1)^2}.$$

Let

$$E_{r,t} = \{(i, j) \in \mathbb{Z}^2 | i \equiv 1, 2, \ldots 2t+1 \bmod r+1\}.$$

Adding to the code C_r^1 the codewords of $E_{r,1}$ gives (for $r \geqslant 2t$) the code $C_{r,1}$ satisfying all the conditions of Theorem 7 and density $D(C_{r,1})$ still approaching $\frac{3}{8}$ when $r \to \infty$, notice that $E_{t,1}$ adds to C_r^1 for growing r and fixed t an amount of codewords (densitywise $\sim (2t+1)/(r+1)$) which asymptotically can be neglected.

Next we show that each translate of A actually contains at least three codewords of C_r^1. The pair of the points $a_1 = (-r-1, 0)$ and $a_2 = (0, -r-1)$ of A is on the same diagonal L_{-r-1}. The pair of the points $a_3 = (-r, 0)$ and $a_4 = (0, -r)$ of A is on the diagonal L_{-r}. Similarly, the pair $a_5 = (0, r)$ and $a_6 = (r, 0)$ is on L_r and the pair $a_7 = (0, r+1)$ and $a_8 = (r+1, 0)$ is on L_{r+1}. So, A checks certain points from two consecutive diagonals L_{-r-1} and L_{-r} and from two other consecutive diagonals (at distance $2r+1$ and $2r+2$ from L_{-r-1}) L_r and L_{r+1}.

Consider a translate $A + (i, j)$. A pair $a_k + (i, j)$ and $a_{k+1} + (i, j)$, $k = 1, 3, 5, 7$, contains at least one codeword of C_r^1 (resp., two codewords) if the diagonal, on which the pair lies, is a translate of S_r (resp., L_0).

Because the code C_r^1 is $(4r+2)$-periodic, it is enough to assume that the leftmost diagonal $L_m = L_{i+j-r-1}$ of $A + (i, j)$—containing the pair $a_1 + (i, j)$ and $a_2 + (i, j)$—is one of the diagonals L_s where $s = 0, 1, \ldots, 4r+1$. We divide these $4r+2$ possibilities into four cases:

(i) Suppose first that the leftmost diagonal L_m is one of the diagonals L_s where $s = 0, 1, \ldots, 2r-1$. Then the pair corresponding to a_1, a_2, i.e., the pair $a_1 + (i, j), a_2 + (i, j)$, together with the pair corresponding to a_3, a_4 give at least three codewords to the translate of A, because L_m is a translate of L_0 and L_{m+1} is a translate of S_r or vice versa.

(ii) If $m = 2r+1, \ldots, 4r$, then the pairs corresponding to a_5, a_6 and a_7, a_8 contain at least three codewords, since $L_{m+(2r+1)}$ and $L_{m+(2r+2)}$ belong to the set $\{L_0 + (4r+2, 0), \ldots, L_{2r} + (4r+2, 0)\}$.

(iii) Let $m = 2r$. Then L_{2r} and $L_{m+(2r+2)} = L_0 + (4r+2, 0)$ are the diagonals containing the pairs corresponding to a_1, a_2 and a_7, a_8 of $A + (i, j)$. These four points are all in the code.

(iv) Let $m = 4r+1$. Then the elements of $L_{m+1} = L_0 + (4r+2, 0)$ and $L_{m+(2r+1)} = L_{2r} + (4r+2, 0)$ are all codewords and hence the points corresponding to a_3, a_4 and a_5, a_6 provide four codewords to the translate of A.

Case $t = 3$: We extend the code C_r^1 by adding all the points of

$$C' = \bigcup_{s \equiv 2r+1, 2r+2, \ldots, 4r+1 \bmod 4r+2} L_s$$

to the code, i.e.,

$$C_r^3 = C_r^1 \cup C'.$$

Now we show that C_r^3 gives to any translate of $A + (i, j)$ at least seven codewords. We can assume again that the leftmost diagonal L_m containing $a_1 + (i, j)$ and $a_2 + (i, j)$ is one of the diagonals L_s where $s = 0, 1, \ldots, 4r+1$.

(a) Assume first that $m = 0, 1, \ldots, 2r-1$. Again the pairs corresponding to a_1, a_2 and a_3, a_4 give at least three codewords as in (i). Moreover, the pairs corresponding to a_5, a_6 and a_7, a_8 lie on the diagonals $L_{m+(2r+1)}$ and $L_{m+(2r+2)}$ which belong to $\{L_{2r+1}, \ldots, L_{4r+1}\}$. Therefore, the points belong to C' and thus all give four more codewords. This amounts to at least seven codewords in the translate of A.

(b) Suppose next that $m = 2r+1, \ldots, 4r$. The pairs corresponding to a_5, a_6 and a_7, a_8 contain, as in (ii), at least three codewords. Moreover, the pairs corresponding to a_1, a_2 and a_3, a_4 belong to C', and hence there are at least seven codewords in the translate of A.

(c) Let $m = 2r$. The pairs corresponding to a_1, a_2 and a_7, a_8 give four codewords as we saw in (iii). Two more codewords come from the pair corresponding to a_3, a_4 which lies on L_{2r+1} and two from the points corresponding to a_5 and a_6 which belong to $L_{4r+1} \subseteq C'$.

(d) Take finally $m = 4r+1$. Again we get four codewords to the translate of A as in (iv). Moreover, the pair corresponding to a_1 and a_2 as well as the pair corresponding to a_7, a_8 belong to C'.

The density

$$D(C_r^3) = \frac{7r^2 + 8r + 2}{2(2r+1)^2}.$$

Combining C_r^3 with $E_{r,3}$ gives the code $C_{r,3}$ which fulfils the requirements of Theorem 7 and whose density tends to $\frac{7}{8}$ as r grows.

Case $t=2$: We modify the code C_3^r in such a way that we delete the codewords in some diagonals from it, i.e,

$$C_r^2 = C_r^3 \Bigg\backslash \left(\bigcup_{s \equiv 0,2,4,\ldots,2r-2 \bmod 4r+2} L_s \right).$$

Again we can assume that the leftmost diagonal L_m of $A+(i,j)$ equals L_s for some $s=0,1,2,\ldots,4r+1$. Let first $m=0,1,\ldots,2r-1$. Then points in the pairs corresponding to a_1, a_2 and a_3, a_4 contain at least one codeword, because L_m or L_{m+1} is a translate of S_r (or L_{2r}). The other four points of the translate of A belong to C_r^2 according to (a). Hence, there are at least five codewords in the translate of A.

Suppose next that $m=2r+1,\ldots,4r$. At least one of the points in the pairs corresponding to a_5, a_6 and a_7, a_8 is a codeword since either L_{m+2r+1} or L_{m+2r+2} is a translate of S_r. By (b), all the other points of A are codewords.

Assume that $m=2r$. The points corresponding to a_i where $i=1,2,\ldots,6$ all belong to the code. If $m=4r+1$, then the points corresponding to $A \backslash \{a_3, a_4\}$ are codewords.

The density

$$D(C_r^2) = \frac{5r^2+7r+2}{2(2r+1)^2}.$$

Combining C_r^2 with $E_{r,2}$ we obtain the claimed code $C_{r,2}$. $\square$

4. Conclusions

In this paper, we consider t-robust $(r, \leqslant \ell)$-identifying codes in the (infinite) king and square grids. We assume that $\ell, r, t > 0$. For the king grid we show that the optimal density in the case $\ell = 1$ equals $(2t+1)/(4r+2)$ when $t \leqslant 2r$. If $\ell = 2$ and $t = 1$, then the optimal density is 1 and $\frac{3}{4}$ for $r=1$ and $r \geqslant 2$, respectively. We also show that for other values of parameters such codes do not exist.

In the square grid we proved that if $t=r=\ell=1$, then the optimal density equals $\frac{2}{3}$. Other results on $\ell=1$ was given in Theorem 6. For $\ell=2$ we provided asymptotically (on r) good codes for all possible values $t=1,2,3$. If $\ell \geqslant 3$, no t-robust $(r, \leqslant \ell)$-identifying codes exist in the square grid.

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